Mid-term Examination

Partial Differential Equations (MATH4220)

(Academic Year 2021/2022, Second Term)

Date: March 07, 2022. **Time allowed:** 08:30 - 10:15.

Recall that, the solution u for 1D heat equation

$$\begin{cases} \partial_t u = \partial_x^2 u, & \text{in } (t,x) \in [0,\infty) \times \mathbb{R}, \\ u_{|t=0} = \phi(x), & \text{for } x \in \mathbb{R}, \end{cases}$$

is given by

$$u(t,x) = \int_{\mathbb{R}} S(t, x-y)\phi(y)\mathrm{d}y, \quad \text{where } S(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

1. What are the types of the following equations.
   (a) (3 points) $\partial_x^2 u - \partial_{xy} u - 3\partial_{yx} u + \partial_y^2 u + 2\partial_y u + 4u = 0$.
   (b) (3 points) $9\partial_x^2 u + 6\partial_{xy} u + \partial_y^2 u + \partial_x u = 0$.
   (c) (4 points) $4\partial_x^2 u - 12\partial_{xy} u + 9\partial_y^2 u + \partial_y u = 0$.
2. Solve the following PDE.
   (a) (5 points) $\partial_x u + 2\partial_y u - 4u = e^{x+y}$ with $u(x,0) = \sin(x^2)$.
   (b) (5 points) $\partial_t u + \frac{3}{2}\partial_x u = 0$ with $u(0,x) = \sin x$.
   (c) (5 points) $x\partial_t u - t\partial_x u = u$, $t, x > 0$ with $u(0,x) = x^2$.
3. (a) (5 points) State the definition of a well-posed PDE problem.
   (b) (5 points) Is the following problem well-posed? Why?

$$\begin{cases} \Delta u(x) = 0, & \text{for } x \in B_1(0), \\ \dfrac{\partial u}{\partial n}(x) = 0, & \text{for } x \in \partial B_1(0). \end{cases}$$

   (c) (10 points) Verifying that $u_n(t,x) = \frac{1}{n}\sin nx e^{-n^2 t}$ solves the following problem

$$\begin{cases} \partial_t v(t,x) = \partial_x^2 v(t,x), & \text{for } (t,x) \in (-\infty,+\infty) \times (0,\pi), \\ v(t,0) = v(t,\pi) = 0, & \text{for } t \in (-\infty,+\infty), \\ v(0,x) = \dfrac{1}{n}\sin nx, & \text{for } x \in [0,\pi], \end{cases}$$

for all positive integer n. How does the energy $E(t,u)=\int_0^\pi |u(t,x)|^2 \mathrm{d}x$ change when $t\to\pm\infty$.

(d) (10 points) Is the following problem well-posed? Why?

$$\begin{cases} \partial_t v(t,x) = \partial_x^2 v(t,x), & \text{for } (t,x)\in(-\infty,0)\times(0,\pi),\\ v(t,0)=v(t,\pi)=0, & \text{for } t\in(-\infty,0),\\ v(0,x)=0, & \text{for } x\in[0,\pi]. \end{cases}$$

4. Derive the solution formula for the following problems.

(a) (10 points) Solve

$$\begin{cases} \partial_t v(t,x) = \partial_x^2 v(t,x) + v(t,x), & \text{for } (t,x)\in\mathbb{R}^+\times\mathbb{R},\\ v(0,x)=\phi(x), & \text{for } t=0, \end{cases}$$

(b) (10 points) Solve

$$\begin{cases} \partial_t v(t,x) = \partial_x^2 v(t,x), & \text{for } (t,x)\in\mathbb{R}^+\times\mathbb{R}^+,\\ v(0,x)=\phi(x), & \text{for } t=0,\\ v(t,0)=0, & \text{for } x=0. \end{cases}$$

(c) (10 points) Solve

$$\begin{cases} \partial_t v(t,x) = \partial_x^2 v(t,x), & \text{for } (t,x)\in\mathbb{R}^+\times\mathbb{R}^+,\\ v(0,x)=\phi(x), & \text{for } t=0,\\ \partial_x v(t,0)=0, & \text{for } x=0. \end{cases}$$

5. (7 points) Suppose u is harmonic in $B_1(0)\setminus\{0\}\subset\mathbb{R}^2$ and satisfies

$$u(x)=o\left(\log(|x|)\right), \quad \text{as } |x|\to 0.$$

Show that u can be defined at 0 so that it is C^2 and harmonic in $B_1(0)$.

6. (8 points) Consider the following exterior Dirichlet problem

$$\begin{cases} \Delta u(x)=0, & \text{for } x\in\mathbb{R}^3\setminus B_1(0),\\ u(x)=0, & \text{for } x\in\partial B_1(0). \end{cases} \tag{1}$$

Show that there exists unique solution u to (1) such that

$$\lim_{r\to\infty}\left(\max_{|x|=r}\int_{B_1(x)} u(\xi)\mathrm{d}\xi\right)=0.$$

************** END OF THE QUESTIONS **************